

Moufang symmetry XII. Reductivity and hidden associativity of infinitesimal Moufang transformations

Eugen Paal

Abstract

It is shown how integrability of the generalized Lie equations of continuous Moufang transformations is related to the reductivity conditions and Sagle-Yamaguti identity.

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1 Introduction

In this paper we proceed explaining the Moufang symmetry. It is shown how integrability of the generalized Lie equations of a local analytic Moufang loop is related to the reductivity conditions and Sagle-Yamaguti identity. The paper can be seen as a continuation of [1, 2, 3, 4, 5].

2 Generalized Lie equations

In [1] the *generalized Lie equations* (GLE) of a local analytic Moufang loop G were found. These read

$$u_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + T_j^\nu(A) \frac{\partial(S_g A)^i}{\partial A^\nu} + P_j^\nu(S_g A) = 0 \quad (2.1a)$$

$$v_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + P_j^\nu(h) \frac{\partial(S_g A)^i}{\partial A^\nu} + T_j^\nu(S_g A) = 0 \quad (2.1b)$$

$$w_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + S_j^\nu(h) \frac{\partial(S_g A)^i}{\partial A^\nu} + S_j^\nu(S_g A) = 0 \quad (2.1c)$$

where gh is the product of g and h , and the auxiliary functions u_j^s , v_j^s , w_j^s and S_j^μ , T_j^μ , $P_j^\mu(g)$ are related with the constraints

$$u_j^s(g) + v_j^s(g) + w_j^s(g) = 0 \quad (2.2)$$

$$S_j^\mu(A) + T_j^\mu(A) + P_j^\mu(A) = 0 \quad (2.3)$$

For $T_g A$ the GLE read

$$v_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + S_j^\nu(A) \frac{\partial(T_g A)^i}{\partial A^\nu} + P_j^\nu(T_g A) = 0 \quad (2.4a)$$

$$u_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + P_j^\nu(h) \frac{\partial(T_g A)^i}{\partial A^\nu} + S_j^\nu(T_g A) = 0 \quad (2.4b)$$

$$w_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + T_j^\nu(h) \frac{\partial(T_g A)^i}{\partial A^\nu} + T_j^\nu(T_g A) = 0 \quad (2.4c)$$

In this paper we inquire integrability of GLE (2.1a–c). and (2.4a–c) Triality [2] considerations are very helpful.

3 Generalized Maurer-Cartan equations and Yamagutian

Recall from [5] that for x in $T_e(G)$ the infinitesimal translations of G are defined by

$$S_x \doteq x^j S_j^\nu(A) \frac{\partial}{\partial A^\nu}, \quad T_x \doteq x^j T_j^\nu(A) \frac{\partial}{\partial A^\nu}, \quad P_x \doteq x^j P_j^\nu(A) \frac{\partial}{\partial A^\nu} \quad \in T_A(\mathfrak{X})$$

with constraint

$$S_x + T_x + P_x = 0$$

Following triality [2] define the Yamagutian $Y(x; y)$ by

$$6Y(x; y) = [S_x, S_y] + [T_x, T_y] + [P_x, P_y]$$

We know from [5] the generalized Maurer-Cartan equations:

$$[S_x, S_y] = S_{[x, y]} - 2[S_x, T_y] \quad (3.1a)$$

$$[T_x, T_y] = T_{[y, x]} - 2[T_x, S_y] \quad (3.1b)$$

$$[S_x, T_y] = [T_x, S_y], \quad \forall x, y \in T_e(G) \quad (3.1c)$$

The latter can be written [2] as follows:

$$[S_x, S_y] = 2Y(x; y) + \frac{1}{3}S_{[x, y]} + \frac{2}{3}T_{[x, y]} \quad (3.2a)$$

$$[S_x, T_y] = -Y(x; y) + \frac{1}{3}S_{[x, y]} - \frac{1}{3}T_{[x, y]} \quad (3.2b)$$

$$[T_x, T_y] = 2Y(x; y) - \frac{2}{3}S_{[x, y]} - \frac{1}{3}T_{[x, y]} \quad (3.2c)$$

4 Reductivity

Define the (secondary) auxiliary functions of G by

$$\begin{aligned} S_{jk}^\mu(A) &\doteq S_k^\nu(A) \frac{\partial S_j^\mu(A)}{\partial A^\nu} - S_j^\nu(g) \frac{\partial S_k^\mu(A)}{\partial A^\nu} \\ T_{jk}^\mu(A) &\doteq T_k^\nu(A) \frac{\partial T_j^\mu(A)}{\partial A^\nu} - T_j^\nu(g) \frac{\partial T_k^\mu(A)}{\partial A^\nu} \\ P_{jk}^\mu(A) &\doteq P_k^\nu(A) \frac{\partial P_j^\mu(A)}{\partial A^\nu} - P_j^\nu(g) \frac{\partial P_k^\mu(A)}{\partial A^\nu} \end{aligned}$$

The Yamaguti functions Y_{jk}^μ are defined by

$$6Y_{jk}^\mu(A) \doteq S_{jk}^\mu(A) + T_{jk}^\mu(A) + P_{jk}^s(A)$$

In [3] we proved

Theorem 4.1. *The integrability conditons of the GLE (2.1a-c) (2.4a-c) read, respectively,*

$$Y_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} = Y_{jk}^\mu(S_g A) \quad (4.1a)$$

$$Y_{jk}^s(g) \frac{\partial (T_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial (T_g A)^\mu}{\partial A^\nu} = Y_{jk}^\mu(T_g A) \quad (4.1b)$$

Consider the first-order approximation of the integrability conditions (4.1a) and (4.1b). We need

Lemma 4.2. *One has*

$$Y_{jk}^\mu = l_{jk}^\mu + \frac{1}{3}C_{jk}^s(S_s^\mu - T_s^\mu) \quad (4.2)$$

Proof. Use formula (3.2b). \square

Introduce the Yamaguti constants Y_{jkl}^i by

$$Y_{jk}^i(g) = Y_{jkl}^i g^l + O(g^2)$$

Then, by defining [1] the third-order associators l_{jkl}^i by

$$l_{jk}^i(g) = l_{jkl}^i g^l + O(g^2)$$

it follows from Lemma 4.2 that

$$Y_{jkl}^i = l_{jkl}^i + \frac{1}{3}C_{jk}^s C_{sl}^i \quad (4.3)$$

Now we can calculate:

$$\begin{aligned} Y_{jk}^i(S_g A) &= Y_{jk}^\mu(A) + \frac{\partial Y_{jk}^\mu(A)}{\partial A^\nu} S_l^\nu(h) g^l + O(g^2) \\ Y_{jk}^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial(S_g A)^\mu}{\partial A^\nu} &= Y_{jkl}^s g^l S_s^\mu(h) + Y_{jk}^\nu(A) \left(\delta_\nu^\mu + \frac{\partial S_l^\mu(A)}{\partial A^\nu} g^l \right) + O(g^2) \end{aligned}$$

Substituting the latter into (4.1a) and compare the coefficients at g^l and replace. By repeating these calculations for (4.1b) we obtain the *reductivity conditions*

$$S_l^\nu(g) \frac{\partial Y_{jk}^\mu(A)}{\partial A^\nu} - Y_{jk}^\nu(A) \frac{\partial S_l^\mu(A)}{\partial A^\nu} = Y_{jkl}^s S_s^\mu(A) \quad (4.4a)$$

$$T_l^\nu(g) \frac{\partial Y_{jk}^\mu(A)}{\partial A^\nu} - Y_{jk}^\nu(A) \frac{\partial T_l^\mu(A)}{\partial A^\nu} = Y_{jkl}^s T_s^\mu(A) \quad (4.4b)$$

Let us rewrite these differential equations as commutation relations.

In the tangent algebra Γ of G define the ternary *Yamaguti brackets* [8] $[\cdot, \cdot, \cdot]$ by

$$[x, y, z]^i \doteq 6Y_{jkl}^i x^j y^k z^l$$

Multiply (4.3) by $6x^j y^k z^l$. Then we have

$$\begin{aligned} [x, y, z] &= 6(x, y, z) + 2[[x, y], z] \\ &= [x[y, z]] - [y[x, z]] + [[x, y], z] \end{aligned}$$

Now from (4.4) it is easy to infer

Theorem 4.3 (reductivity). *The infinitesimal Moufang transformations satisfy the reductivity conditions*

$$6[Y(x; y), S_z] = S_{[x, y, z]} \quad (4.5a)$$

$$6[Y(x; y), T_z] = T_{[x, y, z]} \quad (4.5b)$$

$$6[Y(x; y), P_z] = P_{[x, y, z]} \quad (4.5c)$$

Proof. Commutation relations (4.5a,b) are evident from (4.4a,b) and (4.5c) easily follows by adding (4.5a) and (4.5b). \square

5 Sagle-Yamaguti identity and hidden associativity

Define the triality conjugated translations

$$P^+ \doteq S - T, \quad S^+ \doteq T - P, \quad T^+ \doteq P - S$$

One can easily see the inverse conjugation:

$$3P \doteq T^+ - S^+, \quad 3T \doteq S^+ - T^+, \quad 3P \doteq T^+ - S^+$$

Theorem 5.1 (reductivity). *The infinitesimal Moufang transformations satisfy the reductivity conditions*

$$6[Y(x; y), S_z^+] = S_{[x, y, z]}^+ \quad (5.1a)$$

$$6[Y(x; y), T_z^+] = T_{[x, y, z]}^+ \quad (5.1b)$$

$$6[Y(x; y), P_z^+] = P_{[x, y, z]}^+ \quad (5.1c)$$

Proof. Evident corollary from formulae (4.5). \square

From [2] we know

Proposition 5.2. *Let (S, T) be a Moufang-Mal'tsev pair. Then*

$$6Y(x; y) = [P_x^+, P_y^+] + P_{[x, y]}^+ \quad (5.2a)$$

$$= [T_x^+, T_y^+] + T_{[x, y]}^+ \quad (5.2b)$$

$$= [S_x^+, S_y^+] + S_{[x, y]}^+ \quad (5.2c)$$

for all x, y in M .

Theorem 5.3 (hidden associativity). *The Yamagutian Y of (S, T) obey the commutation relations*

$$6[Y(x; y), Y(z; w)] = Y([x, y, x], w) + Y(z; [x, y, w]) \quad (5.3)$$

if the following Sagle-Yamaguti identity [7, 8] holds:

$$[x, y, [z, w]] = [[x, y, z], w] + [z, [x, y, w]] \quad (5.4)$$

Proof. We calculate the Lie bracket $[Y(x; y), Y(z; w)]$ from the Jacobi identity

$$[[Y(x; y), S_z^+], S_w^+] + [[S_z^+, S_w^+], Y(x; y)] + [[S_w^+, Y(x; y), S_z^+] = 0 \quad (5.5)$$

and formulae (5.2). We have

$$\begin{aligned} 6[[Y(x; y), S_z^+], S_w^+] &= [S_{[x, y, z]}^+, S_w] \\ &= 6Y([x, y, z]; w) - S_{[[x, y, z], w]}^+ \\ 6[[S_z^+, S_w^+], Y(x; y)] &= 36[Y(z; w), Y(x; y)] - 6[S_{[z, w]}^+, Y(x; y)] \\ &= 36[Y(z; w), Y(x; y)] - S_{[x, y, [z, w]]}^+ \\ 6[[S_w^+, Y(x; y), S_z^+] &= 6Y(z; [x, y, w]) - S_{[z, [x, y, w]]}^+ \end{aligned}$$

By substituting these relations into (5.5) we obtain

$$36[Y(x; y), Y(z; w)] - 6Y([x, y, x], w) - 6Y(z; [x, y, w]) = S_{[x, y, [z, w]] - [[x, y, z], w] - [z, [x, y, w]]}^+$$

The latter relation has to be triality invariant. This means that

$$S_a^+ = T_a^+ = P_a^+ \quad (5.6a)$$

$$= 36[Y(x; y), Y(z, w)] - 6Y([x, y, x], w) - 6Y(z; [x, y, w]) \quad (5.6b)$$

where

$$a = [x, y, [z, w]] - [[x, y, z], w] - [z, [x, y, w]]$$

But it easily follows from (5.6a) that

$$S_a = T_a = P_a = 0$$

and due to $a = 0$ commutation relations (5.3) hold. \square

Remark 5.4. A. Sagle [6] and K. Yamaguti proved [7] that the identity (5.4) is equivalent to the Mal'tsev identity. In terms of Yamaguti [8] one can say that the Yamagutian Y is a *generalized representation* of the (tangent) Mal'tsev algebra Γ of G .

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Department of Mathematics
Tallinn University of Technology
Ehitajate tee 5, 19086 Tallinn, Estonia
E-mail: eugen.paal@ttu.ee